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EFFECTS OF TURBULENCE INSTABILITIES ON LASER PROPAGATION

David A. de Wolf

RCA Laboratories

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EFFECTS OF TURBULENCE INSTABILITIES ON LASER PROPAGATION

DAVID A. de WOLF

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13. ABSTRACT

Developments resulting from work on Contract No. F30602-71-C-0356 are reported. In Section 1, some remarks are presented on Furutsu's recent theoretical analysis of irradiance scintillation. In Section 2, we define and compute useful criteria for the resolution of point features on an illuminated target in turbulent air. The results are related to the work on focal-spot areas. The coefficient of the log-amplitude variance in the saturation regime is computed, and we find $\langle \delta\chi^2 \rangle = 0.41(\kappa_m^{7/6} L^3 C_n^2)^{-1/6}$ in Section 3. Finally, the work on power spectra of angle-of-arrival fluctuations is expanded in Section 4 to include a simple interferometer. Two physical cases, simply related to the ray formulas, appear to apply in most practical situations.

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FOREWORD

The Quarterly Report was prepared by RCA Laboratories, Princeton, New Jersey, under Contract No. F30602-72-C-0486. It describes work performed from June 19, 1972 to September 18, 1972 in the Communications Research Laboratory, Dr. K. H. Powers, Director. The principal investigator and project scientist is Dr. D. A. de Wolf.

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The Air Force Program Monitor is Lt. Darryl P. Greenwood.

SUMMARY AND ABSTRACT

Developments resulting from work on Contract No. F30602-71-C-0356 are reported. In Section 1, some remarks are presented on Furutsu's recent theoretical analysis of irradiance scintillation. In Section 2, we define and compare useful criteria for the resolution of point features on an illuminated target in turbulent air. The results are related to the work on focal-spot areas. The coefficient of the log-amplitude variance in the saturation regime is computed, and we find $< \delta\chi^2 > = 0.41 \; (\kappa_m^{7/6} L^3 C_n^2)^{-1/6}$ in Section 3. Finally, the work on power spectra of angle-of-arrival fluctuations is expanded in Section 4 to include a simple interferometer. Two physical cases, simply related to the ray formulas, appear to apply in most practical situations.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

1. SOME REMARKS ON THE FURUTSU THEORY

In a recent Special Report on the XVIIth General Assembly of URSI* held at Warsaw, Poland late in August 1972, we mentioned a discussion on Furutsu's recent work[1]. Furutsu, in work that is exceptionally difficult to read, has derived the following results for beam waves propagating through turbulent air:

(i) The probability density of irradiance I approaches that of a Rice-Nakagami distribution at large propagation distances L.

(ii) Through another type of approximation Furutsu also derives that *the logarithm of I* follows the Rice-Nakagami distribution: interestingly enough this distribution reduces to the log-normal one for points far away from the central beam axis.

In the discussion at URSI we commented on these results, and therefore on parts of Furutsu's theory. Because his work is hard to read, and because Furutsu's results are being quoted indiscriminately, we shall offer these detailed comments on the theory as we see it.

There appear to be some unphysical features of Furutsu's solution. The above-quoted result (ii) contradicts result (i) for points of observation on the beam axis; they cannot both hold simultaneously. Then, as Furutsu makes explicitly clear in the paragraphs following his Eq. (118), the fluctuations tend to zero as the beam wave approaches a plane wave. It is indeed possible that - in some way - the experimental realizations of plane waves are not that close to Furutsu's plane-wave limit of his beam waves, but at any rate, it seems decidedly unphysical that the beam-wave I does fluctuate on the central axis, whereas the plane-wave I does not. We have uncovered a possible source of error in one of Furutsu's approximations which may make (ii) unphysical. In order to discuss it, we have to give a very brief summary of Furutsu's work. It consists of three distinct parts.

1.1 A Formal Expression for P(I)

Through a method involving variational derivatives and cumulants of the random field, Furutsu derives a general expression of the irradiance probability

*International Union of Radio Science (URSI).

density P(I). We quote a form given by Bremmer in a paper presented at URSI in Warsaw[2] for *plane waves:*

$$P(I) = Q(\sigma_{jk}) \left\{ \sigma_{11}^{-1} e^{-(I+\sigma_{01}^2+\sigma_{10}^2)/\sigma_{11}} J_o \left[2i\sigma_{11}^{-1} I^{1/2} \left(\sigma_{01}^2 + \sigma_{10}^2 \right)^{1/2} \right] \right\}$$

$$Q(\sigma_{jk}) = \exp \left\{ \sum_{\substack{j \\ j+k}} \sum_{\substack{k \\ \geq 2}} \frac{\sigma_{jk}}{j!k!} \frac{\partial^{j+k}}{\partial\sigma_{01}^j \partial\sigma_{10}^k} \right\}$$

$$\sigma_{11} = I_o \left[1 - \exp(-2\alpha L)\right]$$

$$\sigma_{01} = \sigma_{10} \equiv I_o^{1/2} \exp(-\alpha L) \tag{1}$$

Here, J_o is a Bessel function; $J_o(ix) = I_o(x)$ in real notation, but we use the symbol I_o for the free-space irradiance and therefore have utilized the Bessel function J_o with imaginary argument. The well-known formal attenuation coefficient of the coherent (average) field is $\alpha \sim k^2 L\ L_o^{5/3}\ C_n^2$. The cumulants σ_{jk} are too complicated to write down but they are given in both [1] and [2]. The summation over j and k excludes j = k = 1. It is this result that yields (i) when ignoring all higher-order cumulants, in which case operator $Q(\sigma_{jk}) \approx 1$. Furutsu and Bremmer stress that the off-diagonal contributions of σ_{jk} (for $j \neq k$) are negligible; this simplifies operator $Q(\sigma_{jk})$ somewhat but not enough to add anything new.

1.2 The Moment Equations

Furutsu therefore abandons this approach and derives the moment equations by his cumulants + variational-derivatives method. We state his equations in the following form: Let $\vec{r}_m = (\vec{\rho}_m, L) = (x_m, y_m, L)$ be the coordinate of a point of observation in the plane z = L. The N-th moment $\langle I^N \rangle$ is derived from a quantity $M(\vec{\rho}_j, \vec{\rho}_k', L) = \langle E(\vec{r}_1)---E(\vec{r}_N)E^*(\vec{r}_1')---E^*(\vec{r}_N') \rangle$ that is a function of L and of the 2N transverse coordinates $\vec{\rho}_1---\vec{\rho}_N, \vec{\rho}_1'---\vec{\rho}_N'$ (symbolized by $\vec{\rho}_j$, $\vec{\rho}_k'$) by setting

$\vec{\rho}'_m = \vec{\rho}_m$, and then allowing all N coordinates $\vec{\rho}_m$ to coincide with $\vec{\rho}_o = 0$, the coordinate of the central axis. The moment equation is

$$\left[\partial/\partial z - (i/2k)D - (k/2)W(\vec{\rho}_j,\vec{\rho}'_k)\right] M(\vec{\rho}_j,\vec{\rho}'_k,z) = 0, \tag{2}$$

completely analogous to the parabolic wave equation in a random dielectric where we have the Laplacian Δ instead of operator D, and $\delta\varepsilon$ instead of iW. The definitions of D and W are

$$D \equiv \sum_{m=1}^{N} (\Delta_m - \Delta'_m)$$

$$W = \sum_{j=1}^{N} \sum_{k=1}^{N} \left[V(\vec{\rho}_k-\vec{\rho}'_j) + V(\vec{\rho}'_k-\vec{\rho}_j) - V(\vec{\rho}_j-\vec{\rho}_k) - V(\vec{\rho}'_j-\vec{\rho}'_k)\right] \tag{3}$$

$$V(\rho) = k\int_0^L dz < \delta\varepsilon(\vec{0},z)\delta\varepsilon(\vec{\rho},z)>$$

Note that W is wholly determined by the plane-wave *mutual-coherence* function (mcf) $V(\rho)$, which - for the Kolmogorov turbulence spectrum of $\delta\varepsilon$ - goes as ρ^2 for $\rho < \ell_o$ (microscale), as $\rho^{5/3}$ for $\ell_o < \rho < L_o$ (macroscale), and saturates as $\rho \to \infty$. No objections to (2) and (3) and no differences with other earlier derivations are noted.

1.3 An Approximate Solution of the Moment Equations

Furutsu finds an approximate solution to (2), namely $M_F(\vec{\rho}_j,\vec{\rho}'_k,z)$ for $\vec{\rho}'_k,\vec{\rho}_j$ (all j,k) by approximating $W(\vec{\rho}_j,\vec{\rho}'_k)$ by $W_F(\vec{\rho}_j,\vec{\rho}'_k)$, the function obtained in (3) by using the ρ^2 form in the $V(\rho)$ for *all* ρ. Furutsu's rationale for so doing is based on the premise that because he wants a solution to (2) for the limiting case that $\vec{\rho}'_k-\vec{\rho}_j \to 0$, for all j,k, he might just as well use a limiting form for W obtained in the limit $\vec{\rho}'_k-\vec{\rho}_j \to 0$. That limiting form is the above-described W_F obtained by utilizing Furutsu's Eq. (68a), and it results in (ii).

We believe the source of possible error, or difference of opinion as to the validity of the result, lies right here. In our view the approximation $W \approx W_F$ yields an $M_F \neq M$. To make this plausible, we recast (2) into an integral equation subject to the boundary condition that $M(\vec{\rho}_j, \vec{\rho}_k', L) \to 1$ as $W \to 0$ (the free-space moments are all unity).

$$M(\vec{\rho}_j, \vec{\rho}_k', L) = 1 - (k/2) \int_0^L dz \left[e^{-(i/2k)(L-z)D} W(\vec{\rho}_j, \vec{\rho}_k') M(\vec{\rho}_j, \vec{\rho}_k', z) \right]_{\vec{\rho}_j = \vec{\rho}_k'} \tag{4}$$

A formal series solution to (4) is obtained through iteration; the resulting series is analogous to that of the Born terms of the integral wave equation.

Furutsu's approximation $W = W_F$ is correct when $D = 0$. This can be seen in general, but also specifically from a simple example: Let $W = \sin^2 z$. Then $W_F = z^2$, obviously the asymptotic form of W as $z \to 0$. If we solve (2) for this simple case we obtain

$$M = \exp\left[-(2z - \sin 2z)/4\right], \quad M_F = \exp(-z^3/3) \tag{5}$$

Clearly, $M \to M_F$ as $z \to 0$, as Furutsu claims it should, and he was presumably guided by such an example. However, when $D \neq 0$, as is the case in (3), matters are entirely different. To start out with, the operator in (4) should work first on WM; only thereafter should one set $\vec{\rho}_j = \vec{\rho}_k'$. The exponential operator containing D brings into play an infinite number of higher-order derivatives of V at the origin of its coordinate. This means that the shape of V (thus also of W) well *away* from the coordinate origin plays a role in determining M! That suggests strongly that $M_F \neq M$ because $W_F(\rho) \neq W(\rho)$ for $\rho > \ell_0$ [3]. We have, in fact, convinced ourselves of this fact by choosing a few simple examples of operators D in the first Born term of (4).

Our conclusion is that Furutsu's result (ii) presumably is not valid for EM waves in turbulent air. It would hold only for a hypothetical random medium governed by a $V(\rho) \propto \rho^2$ for *all* ρ. However, even for such a medium it is difficult to see how one can reconcile result (ii) with (i).

2. RESOLUTION OF FEATURES OF AN ILLUMINATED OBJECT

The resolution of optically visible features of an illuminated object in the turbulent atmosphere has been studied by Fried[4,5]. Fried has computed the MTF (Mutual Transfer Function) of an optical system in the turbulent atmosphere, taking into account a difference between long-term and short-term exposure. His formalism describes the distortion of a point image formed by a lens in turbulent air. We are interested in a somewhat different concept and will therefore adopt another approach.

Consider Fig. 1 where we have sketched a ground-level "eye" observing an object at distance L subtending a geometrical angle $\theta_g = \arctan(\rho_o/L) \approx \rho_o/L$

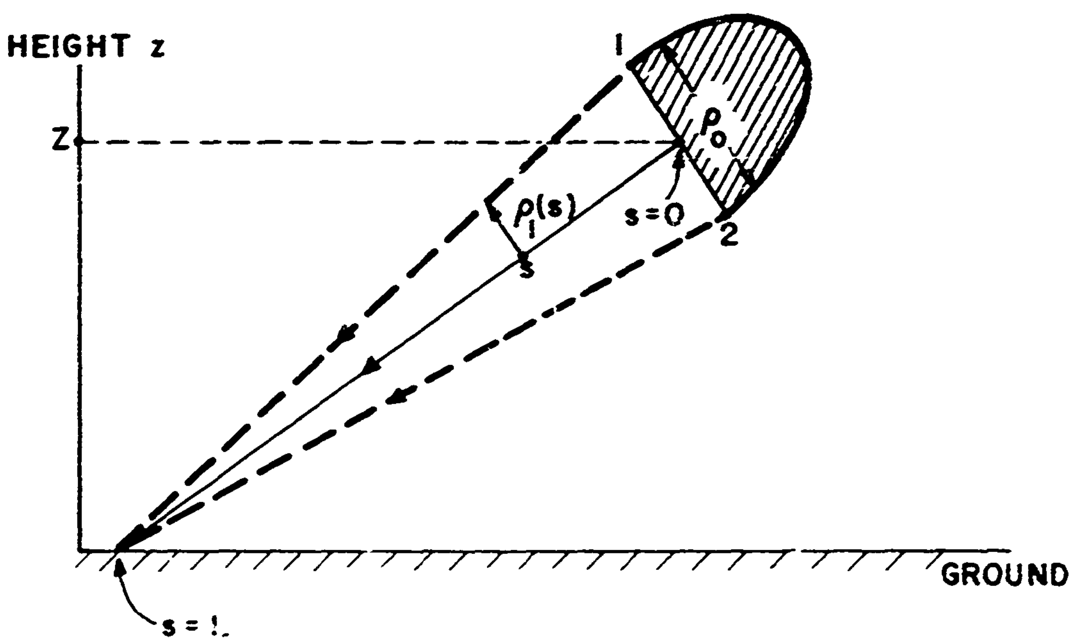

Figure 1. A ground-level observer, and two point features separated by distance ρ_o from each other on an object at range L and height Z.

for small ρ_o/L. We want to know whether or not we can distinguish two point sources on this object separated by angle θ_g in turbulent air, given that we can distinguish between them in free space. To do so, we shall compute the difference $\delta\theta$ in bending of the two rays emanating from each point source which would reach the eye in the absence of turbulence. This corresponds to Fried's short-term exposure. Analogous to long-term exposure would be the distortion which includes refraction of the central ray corresponding to image blurring by random variations of the wavefront tilt. This latter case does not appear very interesting to us because it is always possible to track the total object and thus remove the "wavefront tilt" as Fried terms it. The other case, corresponding presumably to Fried's short-term exposure, appears more interesting because it tells us how features become indistinguishable even when the total object is being tracked. Thus, we are interested in the angle $\delta\theta$ which is appropriately given for all conceivable distances in the atmosphere by the geometrical-optics ray formula,

$$\delta\theta = \frac{1}{2} \int_0^L ds \left\{ \vec{\nabla}_1 \delta\epsilon[\rho_1(s),s] - \vec{\nabla}_2 \delta\epsilon[\rho_2(s),s] \right\}$$

$$\rho_1(s) = (L-s)\rho_o/2L \tag{6}$$

$$\rho_2(s) = -(L-s)\rho_o/2L$$

The derivatives $\vec{\nabla}_1$ and $\vec{\nabla}_2$ are with respect to the transverse coordinates $\rho_1(s)$ and $\rho_2(s)$ of the two rays (sketched in Fig. 1) we are comparing. Clearly $\delta\theta \to 0$ as $\rho_o \to 0$, and other properties expected of this *difference angle* $\delta\theta$ will become apparent later. We shall proceed to calculate the statistics of $\delta\theta$. Analogous to so many other similar quantities in turbulent air. this one too is clearly Gaussian with zero mean, and we need only compute its variance. To describe all the details of this calculation would be extremely tedious as it differs hardly from many previous ones. The notation is given in preceding reports[6]. The technique is always similar. It involves the following steps:

(i) Replace $\vec{\nabla}_T\delta\epsilon(\vec{\rho},s)$ by $-i\vec{K}\delta\tilde{\epsilon}(\vec{k},s)$ through a partial two-dimensional Fourier transform in a plane s = const.

(ii) Use the customary properties of the covariance $\langle\delta\tilde{\epsilon}(\vec{K}_1,s_1)\,\delta\tilde{\epsilon}(\vec{K}_2,s_2)\rangle$ to ignore effects of order L_o/L and to introduce the dielectric turbulence spectrum $\Phi(K)$.

$$\epsilon^2\Phi(K) = 32\pi^3 \times 0.033\, C_n^2 (K^2 + L_o^{-2})^{-11/6} \exp(-K^2/\kappa_m^2) \tag{7}$$

We find

$$<\delta\theta^2> = (4\pi)^{-1}\int_0^L ds\,\epsilon^2(s)\int_0^\infty dK\,K^3\,\Phi(K)\,\{1 - J_0[2K\rho_1(s)]\} \tag{8}$$

where $\rho_1(s)$ is given in (6). As we shall consider fairly small values of $\rho_1(s)$ and because there is a factor K^3 in the above integrand, it can be argued that the factor $(K^2 + L_o^{-2})^{-11/6}$ in (7) may be replaced by $K^{-11/3}$. Inserting (6) and (7) into (8), introducing the auxiliary variable $x = K^2/\kappa_m^2$, and then performing the integration over the new variable x, we obtain:

$$<\delta\theta^2> = 1.3\ \Gamma(1/6)\kappa_m^{1/3}\int_0^L ds\ C_n^2(s)\{1 - M[1/6,1,-\kappa_m^2\rho_o^2(L-s)^2/4L^2]\} \tag{9}$$

The coefficient 1.3 is an approximation of $4 \times 0.033\pi^2$ accurate to about 0.2%, Γ is a gamma function, and $M(a,b,z)$ is a Kummer function defined in (13.1.2) of [7].

2.1 Horizontal Propagation

For horizontal propagation, $C_n^2(s)$ is a constant. Using the above-cited power-series definition of the Kummer function, we expand into a power series and integrate term by term to obtain

$$<\delta\theta^2> = 1.3\ C_n^2\kappa_m^{1/3}L \times \sum_{m=1}^{\infty} \frac{(-1)^{m+1}\Gamma(1/6+m)}{(m!)^2(2m+1)} \cdot \left(\frac{\kappa_m\rho_o}{2}\right)^{2m} \tag{10}$$

This form is useful for small values of $\kappa_m\rho_o$. Thus, when $\rho_o << \ell_o$, i.e. when two distinct features are separated by much less than the microscale, we find a differential refraction with variance.

$$<\delta\theta^2> = 1.3\ C_n^2\kappa_m^{1/3}L \times \frac{\Gamma(7/6)}{12}\ (\kappa_m\rho_o)^2 \quad \text{for } \rho_o << \ell_o \tag{11a}$$

Note that the combination $1.3\ C_n^2\kappa_m^{1/3}L$ is identical to the ratio r_{LB}^2/L^2 where πr_{LB}^2 is the area of the average focal spot in turbulent air minus the vacuum diffraction-limited area as defined in RADC-TR-72-119[6]. On the other hand, when $\rho_o >> \ell_o$ the Kummer function in (9) is negligible and we obtain

$$<\delta\theta^2> = 1.3\, C_n^2 \kappa_m^{1/3} L \times \Gamma(1/6) = \Gamma(1/6) r_{LB}^2/L^2 \quad \text{for } \rho_o >> \ell_o \qquad (11b)$$

Equations (11) provide a basis for deciding whether or not resolution is possible. We suggest - somewhat arbitrarily - that $<\delta\theta^2>$ be compared with $\theta_g^2 \approx \rho_o^2/L^2$. The reason for this criterion is that a ray coming from feature 1 would appear to be coming from feature 2 when these angles are equal, and hence feature 2 would not be indistinguishable from feature 1. However, features 1 and 2 are not confused with each other when $<\delta\theta^2> << \theta_g^2$, i.e., resolution is good. A resolution index of use would be $<(\delta\theta/\theta_g)^2>$. Resolution is good (bad) when this index is much less (more) than unity. From (11) we obtain

$$\begin{aligned} <(\delta\theta/\theta_g)^2> &= \frac{\Gamma(7/6)}{12}(\kappa_m r_{LB})^2 \quad \text{for } \rho_o << \ell_o \\ &= \Gamma(1/6)\ (r_{LB}/\rho_o)^2 \text{ for } \rho_o >> \ell_o \end{aligned} \qquad (12)$$

Consider (12) first of all for *small* $\rho_o (\rho_o << \ell_o)$, i.e. for resolving well within the microscale. It is noteworthy that the criterion is then *independent* of ρ_o! In fact it simply depends upon the ratio of r_{LB} to ℓ_o, i.e. upon the strength of cumulative turbulence along the path, whether or not resolution of small features is possible. Let us recast (12) into a more numerical form:

$$\begin{aligned} <(\delta\theta/\theta_g)^2> &\approx (r_{LB}/0.61\ \ell_o)^2 \quad \text{for } \rho_o << \ell_o \\ &\approx (2.4 r_{LB}/\rho_o)^2 \quad \text{for } \rho_o >> \ell_o \end{aligned} \qquad (13)$$

We have reproduced Fig. 1 of RADC-TR-72-119 as Fig. 2 in this report, so that the user can simply read off r_{LB} for given L and C_n^2 (ℓ_o was chosen to be 6 mm for this graph). Sub-microscale details appear to be unresolvable for $C_n^2 > 10^{-14} m^{-2/3}$ and distances beyond several hundred meters, a hardly surprising

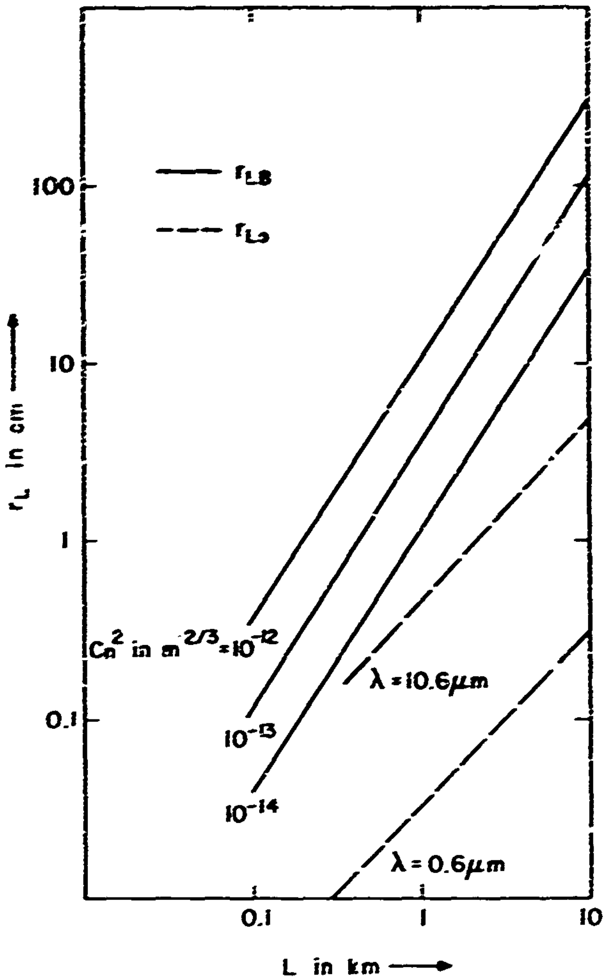

Figure 2. A plot of r_{LB} as a function of range L and turbulence strength C_n^2.

fact and somewhat useless since the microscale is in itself so small. Let us rather consider the other - more practical - case that $\rho_o >> \ell_o$ in which case the second of (13) would hold. In this case resolution is lost for all details that exceed the size $2.4r_{LB}$ which is easily read off Fig. 2. For example, the critical size for distinguishing details at 3-km distance, when $C_n^2 = 10^{-14}m^{-2/3}$, appears to be of the order of 8 cm.

A more formal version of this simple theory can be obtained by utilizing Yura's procedure[8] for computing the mutual-coherence function (mcf) of two delta-function sources at locations 1 and 2 of Fig. 1. It appears sufficient to us, however, to utilize (13) and Fig. 2 for rapid estimates of the amount of detail that can be resolved. In the sense that amplitude effects do not appear to play an important role in the plane-wave and spherical-wave mcf, it would appear that considerations based on (6) suffice.

2.2 Slant-Path Propagation

If the viewed object is at altitude Z, then the development from (8) on is different because $\varepsilon^2(s)$, hence $C_n^2(s)$, is a function of path position. The altitude model of $C_n^2(s)$ is discussed in TR-RADC-72-119[6] in enough detail. We shall adapt the $\varepsilon_o >> l_o$ condition so that the Bessel function in (8) can be ignored. In that case we find

$$\langle \delta\vartheta^2(Z)\rangle = F(Z)\,\langle \delta\vartheta^2(0)\rangle$$
$$F(Z) \equiv ds \int_0^1 (1 + sZ/z_o)^{-2/3}\,(1 + sZ/L_s)^{-2/3}\,\exp(-2sZ/h), \tag{14}$$

where $\langle \delta\vartheta^2(0)\rangle$ is the horizontal-propagation result given by (11b). The Monin-Obukhov length has been divided by 7 to yield L_s; at dawn and dusk $L_s \to \infty$ and at midday (sunny weather) $L_s \approx 1.5$ m. The observation altitude is chosen arbitrarily as $z_o \approx 1$ m, but there is some uncertainty in this choice. There is also considerable arbitrariness in the use of this model for altitudes above several hundred meters. As a result, we shall also restrict ourselves to $Z << h$.

(A) Dawn, dusk approximation, $L_s \to \infty$

In this case, only the first integrand factor of F(Z) deviates from unity, and we obtain

$$F(Z) \sim 3(Z/z_o)^{-2/3}, \quad z_o << Z << h \tag{15a}$$

(B) Midday, sunshine, $L_s \approx 1.5$ m

This is a slightly more complicated case, even for $z_o << Z << h$. Through coordinate transformations, (14) can be recast into

$$F(Z) \approx \left[L_s^2 z_o^2/2Z^3 (L_s - z_o)\right]^{1/3} \int_0^{Z/(L_s - z_o)} dy\; y^{-2/3}\,(1+y)^{-2/3}, \tag{16}$$

which expression contains an integral representation of incomplete beta function. However, the upper bound of the integral is extremely large; hence, we may set it to ∞. We then have the representation of the Kummer function $U(1/3, 2/3, 0)$ (see formula (13.2.5) in ref. [7]), which reduces to a quotient of products of gamma functions. We find

$$F(Z) \sim \frac{\sqrt{\pi}\,\Gamma(1/3)}{\Gamma(5/6)} \times \frac{\ell_s^{2/3}}{z_o^{1/3}(\ell_s - z_o)^{1/3}} \times (Z/z_o)^{-1}, \quad z_o \ll Z \ll h$$

$$\sim 5(Z/z_o)^{-1} \tag{15b}$$

The numerical estimate in (15b) is based on $\ell_s \approx 1.5$ m, $z_o \approx 1$ m. These formulas (15) are intended to be guidelines rather than a rigorous basis. The uncertainties in the $C_n^2(s)$ model are too large for more accurate results. We note a stronger reduction with altitude increase of $\langle\delta\theta^2(Z)\rangle$ compared with an equivalent horizontal-path propagation situation at midday than at dawn or dusk; on the other hand, $C_n^2(0)$ is usually much higher at midday so that conditions for feature resolution are usually better at dawn and dusk, even for objects fairly high in elevation angle.

3. COEFFICIENT OF THE LOG-AMPLITUDE VARIANCE

A new theory of irradiance fluctuations of a plane wave in turbulent air was presented in a previous quarterly report, RADC-TR-72-204[6]. Its major new result was to predict a log-normal distribution of irradiance I in the saturation regime, i.e. where $k^{7/6}L^{11/6}C_n^2$ is not small. More specifically, it defines the saturation regime by the size of the parameter $\sigma_I^2 = \kappa_m^{1/3} kL^2 C_n^2$, which does not differ a great deal from $k^{7/6}L^{11/6}C_n^2$ at optical frequencies, and it predicts for the log-amplitude variance

$$<\delta\chi^2> = (\kappa_m^{7/3}L^3C_n^2)^{-1/6}, \tag{17}$$

when $\sigma_I^2 >> 1$. The right-hand side of (17) may look unfamiliar but it too is not very different from $(k^{7/6}L^{11/6}C_n^2)^{-1/6}$ at optical frequencies. One should note, however, that the predicted combinations of variables in σ_I^2 and in (17) are not restricted to Kolmogorov turbulence statistics. They are more general, and we prefer to use them also because the distinction between them and $k^{7/6}L^{11/6}C_n^2$ cannot be made experimentally to date.

A coefficient of proportionality is missing in (17). The purpose of this section is to estimate it. The starting point is Eq. (A4) in RADC-TR-72-204. This equation yields a real part χ given by

$$\chi = -(k/8\pi^2)\int_0^L dz \int d\&(\vec{K},z)\sin[K^2(L-z)/2k]\exp[-i\vec{K}\cdot\vec{\rho}(z)], \tag{18a}$$

and we use the conservation-of-energy relationship for the variance of χ, namely $<\delta\chi^2> = -<\chi>$, to compute the variance $<\delta\chi^2>$ from (18a). The correlation $<d\&(\vec{K},z)\exp[-i\vec{K}\cdot\vec{\rho}(z)]>$ can be computed to be $(\epsilon^2/4)K^2\Phi(K)(L-z)\exp[-K^2<\rho^2(z)>/4]d^2K$. Substituting this into (18a), utilizing the spectrum $\Phi(K) = 32\gamma C_n^2 K^{-11/3}\exp(-K^2/\kappa_m^2)$, and making the variable changes of K to $x = K^2(L-z)/k$ and z to $y = (L-z)/L$ we obtain

$$<\delta\chi^2> = \frac{1}{2}\gamma k^{7/6}L^{11/6}C_n^2 \int_0^1 dy\, y^{5/6} \int_0^\infty dx\, x^{1/6}\exp[-x\sigma_1^2 f(y)], \quad (18b)$$

where $\gamma = 0.033\pi^2 \approx 0.325$ and $f(y) = \gamma\Gamma(7/6)y(3-2y)$. The last integral in (18b) yields a gamma function $\Gamma(7/6)$, and upon subsequent rearranging of the factors we obtain

$$<\delta\chi^2> = \beta\left(\kappa_m^{7/3}L^3C_n^2\right)^{-1/6}$$

$$\beta = [\gamma\Gamma(7/6)]^{-1/6} \cdot \frac{1}{2} \int_0^1 dy\, y^{-1/3}(3-2y)^{-7/6}. \quad (19)$$

The coefficient β is easily computed numerically, and we finally obtain

$$<\delta\chi^2> = 0.41\left(\kappa_m^{7/3}L^3C_n^2\right)^{-1/6}. \quad (20)$$

This is the final result. In order to facilitate comparison with existing data plotted as measured $<\delta\chi^2>$ vs. observed values of $\sigma_\epsilon^2 = 0.307k^{7/6}L^{11/6}C_n^2$ we rewrite (20) as

$$<\delta\chi^2> \cong 0.34\left(\kappa_m^2 L/k\right)^{-7/36}\left(\sigma_\epsilon^2\right)^{-1/6} \quad (21)$$

This formula indicates a *weak* dependence on frequency and pathlength of the coefficient of $(\sigma_\epsilon^2)^{-1/6}$. If we set $\ell_0 \approx 6$ mm (i.e. $\kappa_m \approx 10^3 m^{-1}$), $\lambda \approx 0.6$ µm (i.e. $k \approx 10^7 m^{-1}$), and choose $L = 1$ km, we find the coefficient of $(\sigma_\epsilon^2)^{-1/6}$ to be 0.14. If we take L = 500 m or set $\lambda = 0.3$ µm, we obtain 0.16. Meyers, et al.[9] present two curves in the saturation regime of $<\delta\chi^2>$ vs. σ_ϵ^2. In our notation they find

$$\begin{aligned} <\delta\chi^2> &= 0.55\ (\sigma_\epsilon^2)^{-0.17} \quad \text{at } \lambda = 0.514\ \mu m \\ &= 0.24\ (\sigma_\epsilon^2)^{-0.17} \quad \text{at } \lambda = 1.15\ \mu m \end{aligned} \quad (22)$$

The agreement of the theory with these data is not unreasonable, but the predicted wavelength dependence of the coefficient is strongly contradicted. However, the 1.15-μm curve is based on only one point in the saturation regime so that perhaps one cannot use these data to predict the wavelength dependence of the coefficient. More recent data taken by Kerr[10] - see his Fig. 7 - indicate a weak wavelength dependence hidden by data spread, and a relationship

$$\langle \delta\chi^2 \rangle \sim 0.3\ (\bar{\sigma}_\epsilon^2)^{-1/6} \qquad \text{(error perhaps 30\%)}, \tag{23}$$

where $\bar{\sigma}_\epsilon^2 = 0.124\ k^{7/6} L^{11/6} C_n^2$ is the parameter for spherical waves. We have not yet computed our coefficient for spherical waves so that we have not checked out this coefficient which contains an appreciable error due to data spread. However, it appears to be the same order of magnitude as for plane-wave data - see Fig. 3 of Gracheva, et al.[11]. For these latter data we infer from the plane-wave plot,

$$\langle \delta\chi^2 \rangle \sim 0.25\ (\sigma_\epsilon^2)^{-1/6} \qquad \text{(error perhaps 20\%)} \tag{24}$$

so that, again, there appears to be reasonable agreement with our results (20) or (21).

4. ANGLE-OF-ARRIVAL POWER SPECTRUM FOR AN INTERFEROMETER

In a previous report (RADC-TR-72-119)[6], the power spectrum of the focal-spot scintillation was shown to be that of the angle of arrival of a ray, and it was computed. In the present report we shall do the same for an interferometer simplified to its essentials. Basically, our interferometer consists of two parallel rays separated by a distance ρ. They are collected by a lens at distance L and focused into a fringe pattern. The "center" of this fringe pattern deviates from the central axis only through mutual ray bending caused by atmospheric irregularities much larger than ρ that are intersected by both rays. All other irregularities give rise to relative bending, i.e., to *distortion* of the fringe pattern. A measure of the distortion is given by the difference $\delta\theta$ of the two angles at which both rays arrive at the lens. This difference is described by the vector.

$$\begin{aligned}\delta\vec{\theta} &= (1/2)\int_0^L dz[\vec{\nabla}_T\delta\varepsilon(\vec{\rho}/2,z) - \vec{\nabla}_T\delta\varepsilon(-\vec{\rho}/2,z)] \\ &= -(i/4\pi^2)\int_0^L dz\int d^2\vec{K}\vec{K}\delta\tilde{\varepsilon}(\vec{K},z)\ \sin[\vec{K}\cdot\vec{\rho}/2]\end{aligned} \tag{25}$$

Each $\vec{\nabla}_T$ refers to the gradient operator working only on the first two transverse coordinates in $\delta\varepsilon$, denoted by $\pm\vec{\rho}/2$. A partial Fourier representation has been introduced in the second line of (25). The vector $\vec{\rho}$ describes magnitude and direction of the separation in the plane of the two rays. Time t is not indicated explicitly in $\delta\vec{\theta}$ or in $\delta\tilde{\varepsilon}(\vec{K},z)$.

Let us define $\Phi_4(K;\tau)$ as the three-dimensional Fourier transform with respect to $\vec{r}_1-\vec{r}_2$ of $\langle\delta\varepsilon(\vec{r}_1,t)\delta\varepsilon^*(\vec{r}_2,t+\tau)\rangle$. Let us also define the autocovariance $R_\theta(\tau) = \langle\delta\vec{\theta}(t)\cdot\delta\vec{\theta}^*(t+\tau)\rangle$. Upon forming this from (25), making the customary approximations for $L_o \ll L$, and performing the usual other manipulations, we find

$$R_\theta(\tau) = (\epsilon^2 L/8\pi^2)\int d^2KK^2\Phi_4(\vec{K};\tau)[1-\cos(\vec{K}\cdot\vec{\rho})] \tag{26}$$

Note that this expression differs from Eq. (28) of RADC-TR-72-119[6] only by an extra factor which describes the interferometer situation. We now apply Favre's hypothesis for $\Phi_4(\vec{K};\tau)$,

$$\Phi_4(K;\tau) = \Phi(K)\ \exp\left[i\vec{K}\cdot\vec{U}_T\tau - 4(K\Delta U\tau)^2/3\right], \tag{27}$$

as set forth in RADC-TR-72-119[6], noting that

$$\epsilon^2\Phi(K) = 32\pi\gamma C_n^2(K^2 + L_o^{-2})^{-11/6}\exp(-K^2/\kappa_m^2) \tag{28}$$

$$\gamma \equiv 0.033\pi^2 \approx 0.325$$

We insert (28) and (27) into (26) and take due note of the fact that there are two fundamental vectors in the problem: (i) the transverse velocity vector $\vec{U}_T$, (ii) the transverse separation vector $\vec{\rho}$. Before writing down the result, we develop notation some more. Let $\vec{\xi} = \vec{\rho}/L_o$ be a normalized separation vector (i.e. in terms of the number of macroscales). Let $\xi_{/\!/}$ be its component in the direction of $\vec{U}_T$, and $\xi_\perp$ be the component perpendicular to $\vec{U}_T$. Let us also introduce the two angular frequencies $\omega_T = U_T/L_o$ and $\Delta\omega = 4\Delta U/L_o\ \sqrt{3}$. The result is then

$$R_\theta(\tau) = 4\gamma C_n^2 L L_o^{-1/3}\int_0^\infty dx\ x(1+x)^{-11/6}\exp\left\{-x\left[(\Delta\omega\tau/2)^2+(\kappa_m L_o)^{-2}\right]\right\}$$

$$\times\left[J_o(\omega_T\tau x^{1/2}) - \frac{1}{2}J_o(p_+\ x^{1/2}) - \frac{1}{2}J_o(p_-\ x^{1/2})\right] \tag{29}$$

$$p_\pm \equiv \left[(\omega_T\tau \pm \xi_{/\!/})^2 + \xi_\perp{}^2\right]$$

We now define the power spectrum $W_\theta(\omega)$ as the integral from $\tau = 0$ to $\tau = \infty$ of $2R_\theta(\tau)\cos\omega\tau$ as in Eq. (32) of RADC-TR-72-119[6]. This yields

$$W_\theta(\omega) = 8\gamma C_n^2 L L_o^{-1/3}[W_\theta(\omega,5/6) - W_\theta(\omega,11/6)]$$

$$W_\theta(\omega,q) = \omega^{-1}\int_0^\infty dy \cos y \int_0^\infty dx(1+x)^{-q}\exp\left\{-x\left[\left(\frac{\Delta\omega}{\omega}\cdot\frac{y}{2}\right)^2+(\kappa_m L_o)^{-2}\right]\right\}$$

$$\times\left\{J_o\left(\frac{\omega_T}{\omega}yx^{1/2}\right) - \frac{1}{2}J_o\left(\frac{\omega_T}{\omega}y_+x^{1/2}\right) - \frac{1}{2}J_o\left(\frac{\omega_T}{\omega}y_-x^{1/2}\right)\right\} \tag{30}$$

$$y_\pm = \left[(y\pm\xi_\parallel\,\omega/\omega_T)^2 + (\xi_\perp\omega/\omega_T)^2\right]^{1/2}.$$

This form for $W_\theta(\omega,q)$ can be simplified somewhat by a coordinate transformation $y \rightarrow -y$ for the third term. After some algebraic work, we obtain

$$W_\theta(\omega,q) = \omega^{-1}\int_0^\infty dy \cos y\int_0^\infty dx(1+x)^{-q}\exp\left\{-x\left[\left(\frac{\Delta\omega}{\omega}\cdot\frac{y}{2}\right)^2+(\kappa_m L_o)^{-2}\right]\right\}$$

$$\times\left\{J_o\left(\frac{\omega_T}{\omega}yx^{1/2}\right) - \cos(\xi_\parallel\omega/\omega_T)J_o\left(\frac{\omega_T}{\omega}\left[y^2+(\xi_\perp\,\omega/\omega_T)^2\right]^{1/2}x^{1/2}\right)\right\} \tag{31}$$

Note that the parameters $\xi_\perp\omega/\omega_T = \rho_\perp\omega/U_T$, and $\xi_\parallel\,\omega/\omega_T = \rho_\parallel\,\omega/U_T$ are the products of ω with a fundamental time $T_\perp = \rho_\perp/U_T$ (and $T_\parallel = \rho_\parallel/U_T$) that describes the time required for air to flow across the interferometer beams.

A third result ensues for the case that $\xi_\perp = 0$, i.e. for the case that $\vec{U}_T//\vec{\rho}$. The obtained spectrum is identical to that for a ray multiplied by the factor $[1 - \cos(\omega T_\parallel)]$ where $T_\parallel = \rho_\parallel/U_T$. It filters out all harmonics of the fundamental beat frequency $T_\parallel^{-1} = U_T/\rho_\parallel$.

From here on, we will set $\xi_\parallel = 0$ to obtain less trivial solutions for air flow in that plane through the central axis that is also perpendicular to the plane of both interferometer rays. It is quite easy to amend the solutions thus obtained to the general case, simply by prefixing the factor $\cos(\omega T_\parallel)$ in front of the result for the second term of (31).

4.1 Frozen Flow (Taylor's Hypothesis)

Let us first solve (31) for the case that $\Delta U = 0$, i.e. no random component of velocity. We obtain,

$$W_\theta(\omega,q) = \omega^{-1}\int_0^\infty dy\cos y\int_0^\infty dx(1+x)^{-q}\exp[-x(\kappa_m L_o)^{-2}]$$
$$\times\left\{J_o\left(\frac{\omega_T}{\omega}yx^{1/2}\right) - J_o\left(\frac{\omega_T}{\omega}[y^2 + (\xi_\perp\omega/\omega_T)^2]^{1/2}x^{1/2}\right)\right\} \tag{32}$$

Following Appendix B of Clifford's work for the y integration[12], we reduce (32) to

$$W_\theta(\omega,q) = \int_{(\omega/\omega_T)^2}^\infty dx(1+x)^{-q}(x\omega_T^2 - \omega^2)^{1/2}\exp(-x/\kappa_m L_o^2)$$
$$\times\left\{1 - \cos(\xi_\perp[x - (\omega/\omega_T)^2]^{1/2})\right\} \tag{33}$$

The following steps are carried out on (33): First substitute $x = z + \omega^2/\omega_T^2$. Then expand $\cos(\xi_T z^{1/2})$ into its power series, substitute $z = t(1+\omega^2/\omega_T^2)$, and apply formula (13.2.5) of ref.[7] to the integrals in t. The ultimate result is

$$W_\theta(\omega,q) = \frac{\sqrt{\pi}}{\omega_T}\left(1 + \omega^2/\omega_T^2\right)^{1/2-q} e^{-\omega^2/\Omega_T^2}\, U\left(1/2, 3/2-q, \frac{\omega^2+\omega_T^2}{\Omega_T^2}\right)$$
$$\times\left\{1 - \sum_{m=0}^\infty \frac{1}{m!}\left[-\frac{\xi_\perp^2}{4}\left(1 + \frac{\omega^2}{\omega_T^2}\right)\right]^m \frac{U[1/2+m, 3/2-q+m, (\omega^2+\omega_T^2)/\Omega_T^2]}{U[1/2, 3/2, (\omega^2+\omega_T^2)/\Omega_T^2]}\right\} \tag{34}$$

quite clearly an extension of our previous result, Eq. (36a) in RADC-TR-72-119[6]. In the general case that $\xi_\parallel \neq 0$, the summation sign in (34) will be preceded by the factor $\cos(\xi_\parallel\,\omega/\omega_T)$. The Kummer function U(a,b,z) is given in formula (13.1.33) of ref.[7]. The power spectrum has three distinct regimes. For

$\omega << \omega_T$ it is flat. For $\omega_T << \omega << \Omega_T$ there is a decrease in $W_\xi(\omega,q)$ with $(\omega/\omega_T)^{1-2q}$ (note that $\Omega_T = \kappa_m L_o \omega_T$). From the first of Eqs. (30) we note that soon after ω increases above ω_T, the $q = 5/6$ term is dominant because it yields a power-spectral decrease as $(\omega/\omega_T)^{-2/3}$ which certainly masks the small correction from the $q = 11/6$ term yielding a $(\omega/\omega_T)^{-8/3}$ decrease. Thus, the power spectrum decreases as $\omega^{-2/3}$ in the regime of greatest interest; $U_T/L_o << \omega << U_T/\ell_o$. When ω exceeds Ω_T, i.e., when we look for spectral components determined by eddies of smaller size than the microscale, we observe a rapid cutoff.

With regard to the interferometer "correction factor" {-----} in (34), it is obvious that $W_\theta(\omega) \to 0$ as $\xi_\perp \to 0$. It is less clear what happens when $\xi_\perp \to \infty$. We know that the interferometer separation becomes infinite, and because eddies are blown across the interferometer in planes perpendicular to that of the interferometer (which are parallel to the central axis) we know that independent realizations intersect each axis. Consequently, the factor {-----}$\to$1, and we obtain twice the single-ray power spectrum. This can be shown more easily by first letting $\Omega_T \to \infty$ in (34), i.e. by considering the asymptote of $\omega << \Omega_T$. In that case (34) reduces to

$$W_\theta(\omega,q) = \frac{\Gamma(q - 1/2)}{\Gamma(q)} \frac{\sqrt{\pi}}{\omega_T} \left(1 + \omega^2/\omega_T^2\right)^{1/2-q}$$

$$\times \left\{ 1 - \frac{2^{3/2-q}}{\Gamma(q - 1/2)} \cos(\xi_\parallel \omega/\omega_T) \xi_\perp^{q-1/2} \left(1 + \frac{\omega^2}{\omega_T^2}\right)^{q/2-1/4} K_{1/2-q} \left[\xi_\perp \left(1 + \frac{\omega^2}{\omega_T^2}\right)^{1/2}\right] \right\} \quad (35)$$

It is quite clear from the asymptotic form of K as $\xi_\perp \to \infty$ that {-----} $\to$ 1. Furthermore, (35) is clearly much handier for numerical purposes than (34), which has more detail in it than we really require in the interesting portion of the power spectrum $(\omega < \Omega_T)$.

4.2 Random Flow (Favre's Hypothesis)

Consider the *general* case, $\Delta U \neq 0$ and $U_T \neq 0$: the case of random Gaussian velocities with a mean $\vec{U}_T$ normal to the propagation direction. We have not been able to perform both integrations in (31). Possibly one of the integrations can be carried out without simplifying the expression. However, it is possible to obtain some closed-form results by abandoning some of the details. We will discuss a few of these.

4.2.1 The $\omega^{-2/3}$ Portion of the Spectrum

In this subsection, we restrict ourselves to $\omega_T \ll \omega \ll \Omega_T$ or $\Delta\omega \ll \omega \ll \Delta\Omega$ ($\Delta\Omega$ is defined similarly as $\Delta\omega$ with Ω_T replacing ω_T). This means that we set $(1 + x)^{-q} \approx x^{-q}$, then set $q = 5/6$ and ignore the $q = 11/6$ contribution, and let $\kappa_m \to \infty$ in (31). We change variable x to $z = t^2$, and obtain

$$W_S(\omega,5/6) = I(0) - \cos(\xi_\ell \omega/\omega_T) I(\xi_\ell),$$

$$I(\xi_\ell) = 2\pi^{-1} \int_0^\infty dy \cos y \int_0^\infty dt\ t^{-2/3} J_0\left\{t[y^2 + (\xi_\ell \omega/\omega_T)^2]^{1/2} \omega_T/\omega\right\} \tag{36}$$

$$\times \exp[-t^2 (y\Delta\omega/2\omega)^2]$$

The following algebraic steps are performed upon (36): (i) multiply the integrand by $1 = \exp[(\xi_\ell \Delta\omega/2\omega_T)^2 t^2] \exp[-(\xi_\ell\ \Delta\omega/2\omega_T)^2 t^2]$ and absorb the last exponential into the last factor of (36); (ii) expand $\exp[\xi_\ell\ \Delta\omega/\omega_T)^2 t^2]$ into its power series; (iii) perform the dt integration, using formula (11.4.28) of ref.[7]; and (iv) perform the term-for-term dy integration using formula (9.6.25) of ref.[7]. The result is

$$I(\xi_\ell) = \frac{\sqrt{\pi}}{\omega} \sum_{m=0}^{\infty} \frac{1}{m!} (4\omega^2 \xi_\ell/\omega_T \Delta\omega)^{1/3} (\xi_\ell \omega/2\omega_T)^m$$

$$\times M(1/6 + m, 1, -\omega_T^2/\Delta\omega^2) K_{m-1/3}(\xi_\ell \omega/\omega_T) \tag{37}$$

where the Kummer function $M(a,b,z)$, also known as ${}_1F_1(a,b,z)$, is defined in (13.1.2) of ref.[7], and $K_\nu(z)$ is a modified Bessel function of fractional order ν. The power spectrum in the $\omega^{-2/3}$ region is thus in general,

$$W_{\pm}(\omega) = 8\pi C_n^2 L_o^{-1/3} W_{\pm}(\omega,5/6)$$

$$
\begin{aligned}
W_{\pm}(\omega,5/6) = {} & \Gamma(1/3)\,\frac{\sqrt{\pi}}{\Delta\omega}\left(\frac{\omega}{\Delta\omega}\right)^{-2/3} M(1/6,1,-\omega_T^2/\Delta\omega^2) \\
& \times \Bigg\{1 - \frac{2\cos(\xi_\parallel \omega/\omega_T)}{\Gamma(1/3)} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\omega\xi_\perp}{2\omega_T}\right)^{n+1/3} K_{n-1/3}(\xi_\perp \omega/\omega_T) \\
& \times \left[M(1/6+n,1,-\omega_T^2/\Delta\omega^2)\big/M(1/6,1,-\omega_T^2/\Delta\omega^2)\right]\Bigg\}
\end{aligned}
\qquad (38)
$$

Note that the first factor of $W_{\pm}(\omega,5/6)$ is in agreement with (vi), page 7, of RADC-TR-72-204[6]; there is only a difference in normalization by a constant factor $8\pi C_n^2 L_o^{-1/3}$ (the factor $4\pi C_n^2 L_o^{-1/3}$ was inadvertently omitted in the ray formula).

Taylor's hypothesis is obtained from (38) in the limit $\Delta\omega \to 0$. It can be seen that (35) results with only one difference: the factor $1 + \omega^2/\omega_T^2$ in (35) is replaced by ω^2/ω_T^2.

The other limit is $\omega_T \to 0$. However, our formulation leading to (38) is really not suitable for dealing with this limit. As $\omega_T \to 0$, there is no longer a preferred direction along which a $\xi_\parallel$ and $\xi_\perp$ can be defined. The factor $\cos(\xi_\parallel \omega/\omega_T)$ in (38) appears moreover to become infinitely oscillatory and hence acts effectively as a zero factor - unless $\xi_\parallel \to 0$ more rapidly than $\omega_T \to 0$. Clearly, the formulation in this fashion is not suitable for small ratios of $\omega_T/\Delta\omega$ or of ω_T/ω. The difficulty lies in the transition from (30) to (31): the formula where the $\cos(\xi_\parallel \omega/\omega_T)$ factor first appears.

Reconsider (30) and rewrite the argument of the second and third Bessel function terms of the integrand as

$$
\begin{aligned}
\frac{\omega}{\omega_T}\, y_{\pm} x^{1/2} &= \left[\left(\xi_\parallel \pm \frac{\omega_T}{\omega} y\right)^2 + \xi_\perp^2\right]^{1/2} x^{1/2} \\
&= \left[1 \pm 2\,\frac{\xi_\parallel}{\xi}\,\frac{\omega_T}{\omega} y + \left(\frac{\omega_T}{\omega}\cdot\frac{y}{\xi}\right)^2\right]^{1/2} \xi x^{1/2}
\end{aligned}
\qquad (39)
$$

Thus, in this limiting case, we find that the curly-bracketed factor of (30) reduces to $\left\{1 - J_0(\xi\sqrt{x}) + O(\omega_T^2/\omega^2)\right\}$, and, if $O(\omega_T^2/\omega^2)$ terms are ignored, we observe that instead of (36) we now obtain after the dy integration:

$$W_{\xi}(\omega,5/6) = I'(0) - I'(\xi),$$

$$I'(\xi) = \frac{\sqrt{\pi}}{\omega}\left(\frac{\omega}{\Delta\omega}\right)^{-2/3}\int_0^{\infty} dt\, t^{-2/3}\, e^{-t}\, J_0(\xi\omega/\Delta\omega\; t^{1/2}) \tag{40a}$$

This formula can be further developed into a power series of $(\omega\xi/\Delta\omega)^2$, but there is no advantage to doing so for numerical work. We simply compute $I'(0)$ to obtain

$$W_{\xi}(\omega,5/6) = \Gamma(1/3)(\sqrt{\pi}/\Delta\omega)(\omega/\Delta\omega)^{-2/3}$$

$$\times\left\{1 - \frac{1}{\Gamma(1/3)}\int_0^{\infty} dt\, t^{-2/3}e^{-t}J_0(\xi\omega/\Delta\omega\; t^{1/2})\right\} \tag{40b}$$

as the leading term in the small $\omega_T/\Delta\omega$ limit when $\omega >> \Delta\omega$.

4.2.2 The Flat Portion of the Spectrum

Here, we restrict ourselves to $\omega << \Delta\omega$, $\omega << \omega_T$. We set $\xi_{II} \to \infty$, replace y by variable y' via $y = y'\,\omega/\omega_T$ (then we drop the prime for convenience) and then let $\omega \to 0$. We obtain

$$W_{\xi}(0,q) = I(0) - \cos(\xi_{\parallel}\omega/\omega_T)I(\xi_{\perp}),$$

$$I(\xi_{\perp}) = \omega_T^{-1}\int_0^{\infty} dy\int_0^{\infty} dx(1+x)^{-q}\, e^{-xy^2(\Delta\omega/2\omega_T)^2}\, J_0\left[\left(y^2 + \xi_{\perp}^2\right)^{1/2}\sqrt{x}\right] \tag{41}$$

Unfortunately, we are not able to reduce this expression greatly. By using the addition theorem

$$J_0((y^2+\xi_\perp^2)^{1/2}\sqrt{x}) = J_0(\xi_\perp\sqrt{x})J_0(y\sqrt{x}) + 2\sum_{m=1}^{\infty}(-1)^m J_{2m}(\xi_\perp\sqrt{x})J_{2m}(y\sqrt{x}) \quad (42)$$

we can recast (41) into the form

$$I(\xi_\perp) = \frac{\sqrt{\pi}}{\Delta u}\, e^{-u_T^2/2\Delta u^2}\Big\{ I_0(u_T^2/2\Delta u^2)F_0(\xi_\perp,q)$$

$$+\, 2\sum_{m=1}^{\infty}(-1)^m I_m(u_T^2/2\Delta u^2)F_m(\xi_\perp,q)\Big\} \quad (43)$$

$$F_m(\xi_\perp,q) \equiv \int_0^{\infty} dx\, x^{-1/2}(1+x)^{-q}J_{2m}(\xi_\perp\sqrt{x})$$

The functions I_m in (43) are modified Bessel functions. The functions $F_m(\xi_\perp,q)$ have to be computed numerically. However, $I(0)$ is easily computed because $F_m(0,q) = 0$ for $m \neq 0$ and $F_0(0,q) = \Gamma(q - 1/2)/\Gamma(q)$. We thus find for the flat part of the spectrum when $\xi_\perp = 0$

$$W_\phi(0) = 8\pi C_n^2 L L_0^{-1/3} \times [\Gamma(1/3)/\Gamma(5/6) - \Gamma(4/3)/\Gamma(11/6)]$$

$$\times \frac{\pi}{\Delta u}\, I_0(u_T^2/2\Delta u^2)\exp(-u_T^2/2\Delta u^2)\,\Big\{1 - \cos(\xi\rho u/u_T)\Big\} \quad (44)$$

We note that one half of the above formula, after also replacing $\{---\}$ by unity, is the generalization of the ray angle-of-arrival power spectrum for $u \ll u_T$, $u \ll \Delta u$. It yields the same limits as (36b) and (38b) with $u \to 0$.

4.2.3 Concluding Remarks

The differences between ray and interferometer formulas, the latter given by (38) and (40) and the former by one half of the leading first terms of each, appear at first sight to be unimportant. The curly-bracketed factors reduce to unity when $\xi_\perp/u_T$ (or $\xi_\parallel/u_T$ in the other case) exceeds unity. Considering the fact that practical interferometers do have $\xi \sim L_0$ i.e., $\xi \sim 1$, we note that $\xi_\parallel u/u_T \geq 1$ even for frequencies u very close to u_T *unless* the wind is blowing in the plane of the interferometer in which case $\xi_\parallel = 0$. Because wind velocities are presumably horizontal, we believe that two cases are distinguishable:

(i) *Vertical interferometer* (no $\xi_\perp$ component, and $\xi \sim 1$): The above remarks pertain and $W_\xi(u)$ is twice the ray $W_\xi(u)$ for all practical purposes.

(ii) *Horizontal interferometer* (no $\xi_\parallel$ component): In this case one can set $\xi_\parallel = 0$ for all practical purposes to obtain twice the ray formula for $W_\xi(u)$ amended by a filter factor $\{1 - \cos(\xi_\perp u/u_T)\}$ in the case that u_T/u is not small, and the factor of (40b) for the other, small case.

It is dubious that more accurate formulas are required in the $u^{-2/3}$ regime, but they can be worked out numerically from the forms given in this section.

REFERENCES

1. K. Furutsu, J. Opt. Soc. Am. 62, 240 (1972).

2. H. Bremmer, "General remarks concerning theories dealing with scattering and diffraction in random media," paper given in Session VI.5 of the August 21-29, 1972 General Assembly of URSI in Warsaw, Poland (to be published at a later date).

3. Bremmer (private communication) has pointed out another possible error in Furutsu's estimate of $V(\rho)$; in his Eq. (57) there may be an error in the bounds of one of the integrals.

4. D. L. Fried, J. Opt. Soc. Am. 56, 1372 (1966).

5. D. L. Fried, J. Opt. Soc. Am. 56, 1380 (1966).

6. D. A. de Wolf, Quarterly and Technical Reports to Rome Air Development Center on Effects of Turbulence Instabilities on Laser Propagation under Contract No. F30602-71-C-0356: RADC-TR-71-249 (October 1971), RADC-TR-72-32 (January 1971), RADC-TR-72-51 (February 1972), RADC-TR-72-119 (April 1972), and RADC-TR-72-204 (July 1972).

7. M. Abramowitz and I. A. Stegun, NBS Handbook of Mathematical Functions, Appl. Math. Series 55, June 1964.

8. H. T. Yura, Appl. Opt. 6, 1399 (1972).

9. G. E. Meyers, M. P. Keister, Jr., and D. L. Fried, J. Opt. Soc. Am. 59, 491 (1969).

10. J. R. Kerr, J. Opt. Soc. Am. 62, 1040 (1972).

11. M. E. Gracheva, A. S. Gurvich, and M. A. Kallistratova, Izv. Vyssh. Ucheb. Zaved. Radiofizika 12, 56 (1970).

12. S. F. Clifford, J. Opt. Soc. Am. 61, 1285 (1971).